2.3 - Linear Equations

Definition: A first-order di□erential equation of the form $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ is a **linear equation** in y (the independent variable). This is put into **standard form** by dividing both sides by $a_1(y)$ to obtain

$$\frac{dy}{dx} + P(x)y = f(x)$$

input function or forcing function

Example: Consider the DE $3\frac{dy}{dx} + 12y = 4$. In standard form, this is $\frac{dy}{dx} + 4y = \frac{4}{3}$.

If we multiply both sides by e^{4x}, ← wha?

we get $e^{4x}\frac{dy}{dx} + 4e^{4x}y = \frac{4}{3}e^{4x}$

Aside: Differentiate $e^{4x}y$ WRT x:

2nd aside: Implicit differentiation

$$x^2y^3 + 4xy^2 = 5$$

$$\frac{d}{dx}(x^2y^3) = 2xy^3 + 3x^2y^2\frac{dy}{dx}$$

Chain Rule

$$\frac{d}{dx}\left[e^{4x}y\right] = e^{4x}\frac{dy}{dx} + 4e^{4x}y$$

$$\frac{d}{dx}\left[e^{4x}y\right] = \frac{4}{3}e^{4x}$$

Integrate WRT x: $e^{4x}y = \frac{1}{3}e^{4x} + C$

$$y = \frac{1}{3} + Ce^{-4x}$$

Where did e^{4x} come from?

For the eqn in standard form

$$\frac{dy}{dx} + P(x)y = f(x), \qquad (*)$$

Suppose we have some function,

μ-mu

say $\mu(x)$ such that when we multiply (*) by μ, we get

$$\mu\frac{dy}{dx} + \boxed{\mu P}\, y = \mu f(x)$$

If it happens that $\boxed{\frac{d\mu}{dx} = \mu P}$, then

the LHS becomes

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu f(x)$$

$$f\, g' + f'\, g$$

Then we have $\frac{d}{dx}(\mu y) = \mu f(x)$

To find μ, solve the separable equation

$$\frac{d\mu}{dx} = \mu P(x) \Rightarrow \frac{d\mu}{\mu} = P(x)\, dx$$

$$\ln|\mu| = \int P(x)\, dx + C_1 \Rightarrow \boxed{\mu = c\, e^{\int P(x) dx}}$$

No constant needed

Example: Find the general solution of the given di□erential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

$y' + 2xy = x^3$ – std form

linear with $P(x) = 2x \Rightarrow \mu = e^{\int P(x) dx}$ becomes

$$\mu = e^{\int 2x\, dx} \Rightarrow \mu = e^{x^2}$$

$$e^{x^2} y' + 2x e^{x^2} y = e^{x^2} x^3$$

$$\frac{d}{dx}(e^{x^2} y) = x^3 e^{x^2}$$

$$e^{x^2} y = \int x^3 e^{x^2} dx$$

$u = x^2$ $\quad dv = x e^{x^2} dx$

$du = 2x\, dx$ $\quad v = \frac{1}{2} e^{x^2}$

$$e^{x^2} y = \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx$$

$$e^{x^2}y = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

$$y = \frac{1}{2}x^2 - \frac{1}{2} + Ce^{-x^2}$$

This is a general solution because it contains C.

Interval: $(-\infty, \infty)$

Since $Ce^{-x^2} \Rightarrow 0$ as $x \Rightarrow \infty$, it's called a transient term.

$$\mu = e^{\int P(x)dx}$$

Example: $(1+x)\frac{dy}{dx} - xy = \overbrace{x + x^2}^{x(1+x)}$

std form: $\frac{dy}{dx} - \frac{x}{1+x}y = x$

$P(x) = -\frac{x}{1+x} \Rightarrow \int -\frac{x}{1+x}dx$

$$\begin{array}{r} 1 - \frac{1}{x+1} \\ x+1\overline{)\,x\qquad} \\ \underline{-x-1} \end{array}$$

$$\int P(x)dx = \int\left(\frac{1}{x+1} - 1\right)dx$$

OR $\frac{x+1-1}{1+x} = \frac{1+x}{1+x} - \frac{1}{1+x}$

$$\mu = e^{\int P(x)dx} = e^{\ln(x+1) - x}$$

assuming $x+1 > 0$ (interval)

$$\mu = (1+x)e^{-x}$$

$$\frac{d}{dx}\left[(1+x)e^{-x}y\right] = (x^2+x)e^{-x}$$

$$(1+x)e^{-x}y = \int (x^2+x)e^{-x}dx$$

$u = x^2+x \qquad dv = e^{-x}dx$

$du = (2x+1)dx \qquad v = -e^{-x}$

$$(1+x)e^{-x}y = -(x^2+x)e^{-x} + \int (2x+1)e^{-x}\,dx$$

After another IBP:

$$(1+x)e^{-x}y = -(x^2+x)e^{-x} - (1+2x)e^{-x} - 2e^{-x} + C$$

$$y = -x - \frac{1+2x}{1+x} - \frac{2}{1+x} + \frac{Ce^{x}}{1+x}$$

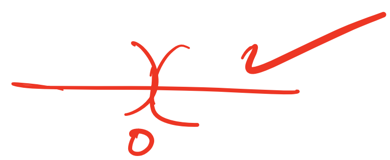
$$y = -x - \frac{2x+3}{1+x} + \frac{Ce^{x}}{1+x}$$

No transient terms.

Interval: $(-1, \infty)$

$\cancel{(-\infty, -1) \cup (-1, \infty)}$

Example: $xy' + (1+x)y = e^{-x}\sin 2x$



$$y' + \frac{1+x}{x}y = \frac{e^{-x}}{x}\sin 2x$$

$$P(x) = \frac{1}{x}+1 \Rightarrow \mu = e^{\int(\frac{1}{x}+1)dx}$$

$\int \frac{1}{x}dx = \ln|x|+C$

$$\mu = e^{\ln x + x}$$

assuming $x>0$

$$\mu = xe^{x}$$

$$\frac{d}{dx}(xe^{x}y) = \sin 2x$$

$$xe^{x}y = -\frac{1}{2}\cos 2x + C$$

$$y = -\frac{1}{2x}e^{-x}\cos 2x + \frac{C}{x}e^{-x}$$

Interval: $(0,\infty)$

It's all transient

Example: $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

input (indep): t
output (dep): P

$$\frac{dP}{dt} + (2t-1)P = 4t-2$$

$$\mu = e^{\int (2t-1)\,dt} = e^{t^2-t}$$

$$\frac{d}{dt}\left(e^{t^2-t}P\right) = (4t-2)e^{t^2-t}$$

$u = t^2 - t \Rightarrow du = (2t-1)dt$

$$e^{t^2-t}P = 2e^{t^2-t} + C$$

$$\boxed{P = 2 + Ce^{t-t^2}}$$

Interval: $(-\infty, \infty)$
transient term: Ce^{t-t^2}

Example: $(x^2-1)\dfrac{dy}{dx} + 2y = (x+1)^2$

Example: Solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

$$(1+x^2)\frac{dy}{dx} + 2xy = f(x), \qquad y(0) = 0, \text{ where}$$

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x & x \ge 1 \end{cases}$$

$x = 0$

$0 \le x < 1$

$$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{x}{1+x^2}$$

$x \ge 1$

$$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{-x}{1+x^2}$$

$$\mu = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)}$$

$$\frac{d}{dx}\left[(1+x^2)y\right] = x$$

$$(1+x^2)y = \tfrac{1}{2}x^2 + C_1$$

$$y = \frac{x^2}{2(1+x^2)} + \frac{C_1}{1+x^2}$$

$$\frac{dy}{dx}\left[(1+x^2)y\right] = -x$$

$$y = -\frac{x^2}{2(1+x^2)} + \frac{C_2}{1+x^2}$$

$y(0) = 0$

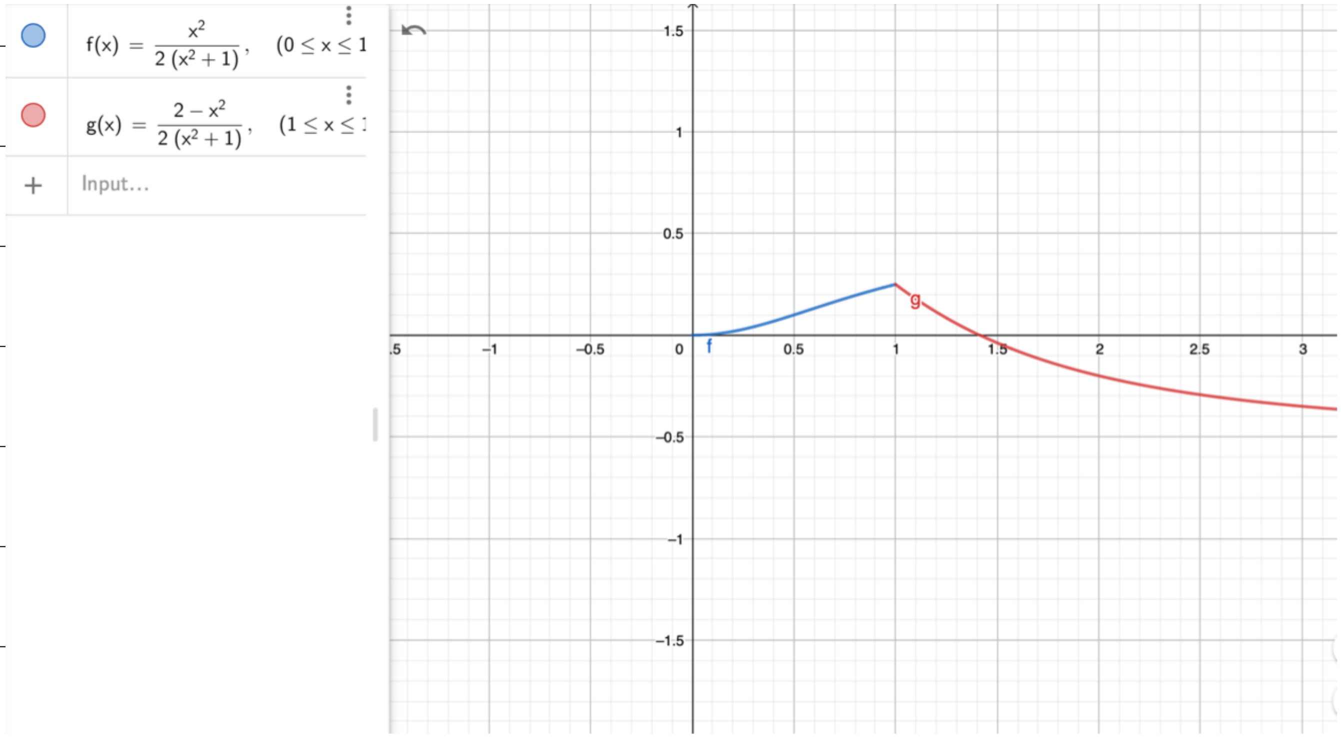
$C_1 = 0$

$$y = \frac{x^2}{2(1+x^2)}, \quad 0 \le x < 1$$

For y to be cont at $x = 1$, $\lim_{x\to 1} y(x) = y(1)$ and for the limit to exist at 1, $\lim_{x\to 1^-} y(x) = \lim_{x\to 1^+} y(x)$

$$\frac{1}{4} = -\frac{1}{4} + \frac{C_2}{2} \Rightarrow C_2 = 1$$

$$\boxed{y(x) = \begin{cases} \dfrac{x^2}{2(1+x^2)}, & 0 \le x < 1 \\[2ex] \dfrac{2-x^2}{2(1+x^2)}, & x \ge 1 \end{cases}}$$



$f(x) = \frac{x^2}{2(x^2+1)}, \quad (0 \le x \le 1$
$g(x) = \frac{2-x^2}{2(x^2+1)}, \quad (1 \le x \le$
Input...
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0.5
1
1.5
2
2.5
3
f
g